* Chapter 4

## 4.1 The Taylor Series:

Taylor Series: Approximate method that can be used to predict a function ($f(x)$) value at a certain point ($x$).

- For example, to approximate function $f(x)$ about a point ($x_i$), Mathematically, thus:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)}{n!} h^n$$

$f^{(n)}(x_i)$, derivate (n) of $f(x)$ at $x = x_i$

$h$: step size

$x_i$: Base point

$n$: # of terms

* Special cases

① $n=0$ ( Zero-order approximation)

$$f(x) \cong \frac{f^{(0)}(x_i)}{\underset{1=}{0!}} \underset{\overset{\shortparallel}{1}}{h^0} = f(x_i)$$

$$, f^{(0)}(x_i) = f(x_i)$$

② $n=1$ ( First-order approx.)

$$f(x) = \sum_{n=0}^{1} \frac{f^{(n)}(x_i)}{n!} h^n = \frac{f^{(0)}(x_i)}{0!} h^0 + \frac{f^{(1)}(x_i)}{1!} h^{(1)}$$

$$= f(x_i) + f'(x_i)\, h$$

③ $n=2$ (second order approx.)

$$f(x) \cong \sum_{n=0}^{2} \frac{f^{(n)}(x_i)}{n!} h^n \cong \frac{f^{(0)}(x_i)}{0!} h^0 + \frac{f^{(1)}(x_i)}{1!} h^{(1)} + \frac{f^{(2)}(x_i)}{2!} h^2$$

$$f(x) \cong f(x_i) + f'(x_i) h + \frac{f''(x_i)}{2} h^2$$

④ nth-order approximation (any $n$)

$$f(x) = f(x_i) + f'(x_i) h + \frac{f''(x_i)}{2} h^2 + \dots + \frac{f^{(n)}(x_i)}{n!} h^n$$

Notes ① If infinite number of terms were used in TS ($n=\infty$), we will get exact value of $f(x)$

② Usually, we use finite number of terms to find approximate values of $f(x)$

③ Difference between exact value and approximate value is called Remainder ($R_n$) or Truncation error

$$R_n = \frac{f^{(n+1)}(x_i)}{(n+1)!} h^{(n+1)}$$

④ TS accuracy depends on step size ($h$) and $n$ (# of terms)

$h \uparrow$ accuracy $\downarrow$, $n \uparrow$ accuracy $\uparrow$

$R_n \Rightarrow$ $h \uparrow$ $R_n \uparrow$, $n \uparrow$ $R_n \downarrow$

Example $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$

Using zero-order to Fourth-order Taylor series approximation

Find ① $f(1)$, given that $x_i = 0$, $h = 1$

② $R_4$ (remainder $n=4$, or 4th order remainder)

Solution

① * $n=0$

$f(x) \cong f(x_i)$

$\Rightarrow f(1) \cong f(0) \cong 1.2$

$\boxed{f(1) \cong 1.2}$

* $n=1$

$f(x) = f(x_i) + f'(x_i)h$

$\Rightarrow f(1) = f(0) + f'(0)(1)$ , $f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$

$f'(0) = -0.25$

$= 1.2 + (-0.25)(1)$

$\boxed{f(1) = 0.95}$

* $n=2$

$f(x) \cong f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2}$

$f(1) \cong f(0) + f'(0)(1) + \frac{f''(0)(1)^2}{2}$ , $f''(x) = 1.2x^2 - 0.9x - 1$

$f''(0) = -1$

$\cong 1.2 + (-0.25)(1) + \frac{(-1)(1)^2}{2}$

$\boxed{f(1) = 0.45}$

* $n=3$

$f(x) \cong f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2} + \frac{f'''(x_i)h^3}{3!}$ , $3! = 3\times2\times1 = 6$

$f(1) \cong f(0) + f'(0)(1) + \frac{f''(0)(1)^2}{2} + \frac{f'''(0)(1)^3}{6}$

$\cong 0.45 + \frac{-0.9}{6}(1)^3$ , $f'''(x) = -2.4x - 0.9$

$\boxed{f(1) \cong 0.3}$ $f'''(0) = -0.9$

* $\underline{n=4}$

$$f(x) = \cdots + \frac{f^{(4)}(x_i)\, h^4}{4!} = 0.3 + \frac{f^{(4)}(0)\,(1)^4}{24} \quad , \; 4! = 4\times3\times2\times1 = 24$$

$$f^{(4)}(x) = -2.4$$

$$f(1) = 0.3 + \frac{(-2.4)}{24} \Rightarrow \boxed{f(1) = 0.2}$$

② $\underline{R_4}$ ? $\quad , \; R_n = \frac{f^{(n+1)}(x_i)}{(n+1)!} h^{n+1}$

$$R_4 = \frac{f^{(5)}(0)\, h^5}{5!} = 0 \quad , \quad R_4 = 0$$

$$f^{(5)}(0) = 0$$

<u>Exact value</u> of $f(1) = ?$

$$f(1) = (0.1)(1)^4 - 0.15(1)^3 - 0.5(1)^2 - 0.25(1) + 1.2$$

$$= 0.2$$

<u>How</u> to know If TS gets more accurate as $n\uparrow$

True Error $(E_t) = |\text{Exact value} - \text{approximate value}|$

Relative true error $(\epsilon_t) = \left| \frac{\text{Exact} - \text{approx}}{\text{Exact}} \right| \cdot 100\%$

Approximate Error $(E_a) = |\text{Present approximation} - \text{Previous approximation}|$

Relative approx. error $(\epsilon_a) = \left| \frac{\text{Present approx.} - \text{Previous approx.}}{\text{Present approx.}} \right| 100\%$

**Example** For the previous example, find $E_t$, $\epsilon_t$, $E_a$ and $\epsilon_a$ for each approximation. Knowing that exact (true) value of $f(1) = 0.2$

**Solution**

| $n$ | $f(1)$, TS | $E_t$ | $\epsilon_t$ | $E_a$ | $\epsilon_a$ |
|---|---|---|---|---|---|
| 0 | 1.2 | 1 | 500% | N/A | N/A |
| 1 | 0.95 | 0.75 | 375% | $+0.25$ | $+26.3\%$ |
| 2 | 0.45 | 0.25 | 125% | 0.5 | 111.1% |
| 3 | 0.3 | 0.1 | 50% | 0.15 | 50% |
| 4 | 0.2 | 0.0 | 0% | 0.1 | 50% |

$n=0$

$E_t = |\text{True} - \text{approx}| = |0.2 - 1.2| = 1$

$$\epsilon_t = \left| \frac{\text{true} - \text{approx}}{\text{true}} \right| 100\% = \left| \frac{0.2 - 1.2}{0.2} \right| 100\% = 500\%$$

$E_a = |\text{Present approximation} - \text{previous approximation}|$

$= 1.2 - (*) \Rightarrow$ N/A "Not available"

$$\epsilon_a = \left| \frac{\text{Present} - \text{previous}}{\text{previous}} \right| 100\% = \text{N/A}$$ "not available"

$\underline{n=1}$

$$E_t = |0.2 - 0.95| = 0.75$$

$$\epsilon_t = \left|\frac{0.2 - 0.95}{0.2}\right| = 375\ \%$$

$$E_a = |\text{Present} - \text{previous}| = |0.95 - 1.2| = 0.25$$

$$\epsilon_a = \left|\frac{\overset{E_a}{\text{Present} - \text{previous}}}{\text{Present}}\right| 100\ \% = \left|\frac{0.95 - 1.2}{0.95}\right| = 26.3\ \%$$

$\underline{n=2}$

$$E_t = |0.2 - 0.45| = 0.25$$

$$\epsilon_t = \left|\frac{0.25}{0.2}\right| = 125\ \%$$

$$E_a = |0.45 - 0.95| = 0.5$$

$$\epsilon_a = \left|\frac{0.5}{0.45}\right| 100\ \% = 111.1\ \%$$

$\underline{n=3}$

$$E_t = |0.2 - 0.1| = 0.1$$

$$\epsilon_t = \left|\frac{0.1}{0.2}\right| 100\ \% = 50\ \%$$

$$E_a = |0.3 - 0.45| = 0.15$$

$$\epsilon_a = \left|\frac{0.15}{0.3}\right| 100\ \% = 50\ \%$$

$\underline{n=4}$

$$E_t = |0.2 - 0.2| = 0$$

$$\epsilon_t = \left|\frac{0}{0.2}\right| 100\ \% = 0\ \%$$

$$E_a = |0.2 - 0.3| = 0.1$$

$$\epsilon_a = \left|\frac{0.1}{0.2}\right| 100\ \% = 50\ \%$$

* A note of previous example

Find 5th and 100th approximation using Taylor Series of $f(1)$, given $x_i = 0$, $h = 1$

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5th-order approximation

$$f(x) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2} + \frac{f'''(x_i)h^3}{3!} + \frac{f^{(4)}(x_i)h^4}{4!}$$

$$+ \frac{f^{(5)}(x_i)h^5}{5!} \dots$$

$$f(1) = 0.2 + \frac{f^{(5)}(0)(1)^5}{5!} \quad , \quad f^{(5)}(0) = 0$$

$$\Rightarrow f(1) = 0.2$$

100th order

$$f(x) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2} + \dots + \frac{f^{(100)}(x_i)h^{100}}{100!}$$

$f(1) = 0.2$ ! why? $f^{(5)}$ and above is zero

Therefore, for a polynomial with degree $(m)$. When we use TS of order $(n \geq m) \Rightarrow f(x) =$ Exact value! Always!